Stress Analysis for Linear Viscoelastic Cylinders

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An alternative method is developed to find the stress and strain distributions in a linear viscoelastic cylinder. The problem of a hollow cylinder with moving inner boundary is presented. The cylinder is encased in and bonded to an elastic casing, and subjected to a uniform internal pressure. It is found that this particular problem can be reduced to a typical Volterra integral equation of the second kind. Solutions can be obtained either analytically by a method of successive approximations, or numerically by means of a finite-difference technique.

1. Introduction

THE determination of the stress distributions in a viscoelastic cylinder is of notable technical interest in solid propellant stress analysis. It has been one of the most extensively studied topics in the field of viscoelasticity. A short review of previous attempts to solve this problem was presented by Rogers and Lee.[1]

The problem of a pressurized viscoelastic hollow cylinder with outer surface reinforced by an elastic cylindrical casing was considered by Huang, Lee, and Rogers.[2] The inner surface was ablating and the cylinder was considered to be viscoelastically compressible. In general, the problem was reduced to two coupled integral equations for solving two unknowns. Only numerical results were possible. Similar cases have also been considered by different workers, such as Ting.[3]

It is the purpose of the present investigation to introduce a much simpler alternative procedure for the analysis. We find that the general cylinder problem can be solved in terms of two unknown functions of t, which should be determined from the given boundary conditions. The problem of a reinforced cylinder with an ablating inner surface can then be reduced to only a single integral equation. It is a Volterra integral equation of the second kind. Solution can be found either numerically by finite difference techniques or analytically by a method of successive approximations.

2. Cylinder Problem

Let us consider an annular cylinder with axial symmetry. The loading is limited to axially uniform pressures on the inner and outer surfaces. We consider that the cylinder is also spinning about its axis with time-dependent angular velocity $\omega(t)$. The cylinder is assumed to be infinitely long and is then considered as a quasi-static plane strain problem.

The only nonvanishing displacement is the radial displacement $u(r,t)$. The infinitesimal radial and tangential strains are

$$e_r = \partial u/\partial r \tag{1}$$

$$e_\theta = u/r \tag{2}$$

where r and θ are the polar coordinates and t is the time.

In terms of the radial displacement u, the volume change per unit original volume v can be expressed in the form

$$\partial u/\partial r + u/r = v \tag{3}$$

where in general

$$v = v(r,t) \tag{4}$$

The differential Eq. (3) can then be solved by direct integration

$$(1/r)(\partial/\partial r)(ru) = v$$

And,

$$u(r,t) = A(t)/2r + (1/r)\int^r sv(s,t)ds \tag{5}$$

where $A(t)/2$ is an arbitrary function of t and s an integration parameter. By strain-displacement relations (1) and (2), we get

$$e_r = -A(t)/2r^2 + v(r,t) - (1/r^2)\int^r sv(s,t)ds \tag{6}$$

$$e_\theta = A(t)/2r^2 + (1/r^2)\int^r sv(s,t)ds \tag{7}$$

The stress-strain relations for linear compressible isotropic viscoelastic materials are

$$\sigma_{r,\theta} = \int_{0^-}^t J(t-\tau)\frac{\partial}{\partial\tau}v(\tau)d\tau + \int_{0^-}^t K(t-\tau)\frac{\partial}{\partial\tau}e_{r,\theta}(\tau)d\tau \tag{8}$$

$$\sigma_z = \int_{0^-}^t J(t-\tau)\frac{\partial}{\partial\tau}v(\tau)d\tau \tag{9}$$

where σ_r, σ_θ, and σ_z are respectively the radial, tangential, and axial stress components. $J(t)$ and $K(t)$ are relaxation functions. The body is assumed to be undisturbed prior to $t = 0$. The only nontrivial equation of equilibrium has the form

$$\partial\sigma_r/\partial r + (1/r)(\sigma_r - \sigma_\theta) + \rho\omega^2(t)r = 0 \tag{10}$$

where ρ is the density and $\omega(t)$ the spinning velocity. Now,

$$\sigma_r - \sigma_\theta = \int_{0^-}^t K(t-\tau)\frac{\partial}{\partial\tau}\left[-\frac{A(\tau)}{r^2} + v(r,t) - \frac{2}{r^2}\int^r sv(s,t)ds\right]d\tau \tag{11}$$

$$\frac{\partial\sigma_r}{\partial r} = \int_{0^-}^t \left\{J(t-\tau)\frac{\partial^2}{\partial r\partial\tau}v(r,\tau) + K(t-\tau)\times \frac{\partial}{\partial\tau}\left[\frac{A(\tau)}{r^3} + \frac{\partial}{\partial r}v(r,\tau) + \frac{2}{r^3}\int^r sv(s,\tau)ds - \frac{v(r,\tau)}{r}\right]\right\}d\tau \tag{12}$$

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Substituting the expressions (11) and (12) into (10), we find

$$\int_{0^-}^{t} [J(t-\tau) + K(t-\tau)] \frac{\partial^2}{\partial r \partial \tau} v(r,\tau) d\tau + \rho\omega^2(t) r = 0 \tag{13}$$

For convenience, we introduce

$$\Omega(t) = \int_{0^-}^{t} H(t-\tau) \frac{\partial}{\partial \tau} \omega^2(\tau) d\tau \tag{14}$$

where auxiliary material function $H(t)$ is defined by

$$\int_{0^-}^{t} [J(t-\tau) + K(t-\tau)] \frac{\partial}{\partial \tau} H(\tau)\, d\tau = 1 \tag{15}$$

Equation (13) can then be inverted in the form

$$v(r,t) = v_0(t) - \tfrac{1}{2}\rho\Omega(t) r^2 \tag{16}$$

where $v_0(t)$ is an arbitrary function of t.

Consequently, expressions of e_r and e_θ are reduced to

$$e_r = v_0(t)/2 - A(t)/2r^2 - \tfrac{3}{8}\rho r^2 \Omega(t) \tag{17}$$

$$e_\theta = v_0(t)/2 + A(t)/2r^2 - \tfrac{1}{8}\rho r^2 \Omega(t) \tag{18}$$

The corresponding stress components can be written as

$$\sigma_r = \int_{0^-}^{t} \left[J(t-\tau) + \frac{1}{2} K(t-\tau) \right] \frac{\partial}{\partial \tau} v_0(\tau) d\tau - \frac{1}{2r^2} \int_{0^-}^{t} K(t-\tau) \frac{\partial}{\partial \tau} A(\tau) d\tau - \frac{1}{2} \rho r^2 \times \left\{ \omega^2(t) - \frac{1}{4} \int_{0^-}^{t} K(t-\tau) \frac{\partial \Omega(\tau)}{\partial \tau} d\tau \right\} \tag{19}$$

$$\sigma_\theta = \int_{0^-}^{t} \left[J(t-\tau) + \frac{1}{2} K(t-\tau) \right] \frac{\partial}{\partial \tau} v_0(\tau) d\tau + \frac{1}{2r^2} \int_{0^-}^{t} K(t-\tau) \frac{\partial}{\partial \tau} A(\tau) d\tau - \frac{1}{2} \rho r^2 \times \left\{ \omega^2(t) - \frac{3}{4} \int_{0^-}^{t} K(t-\tau) \frac{\partial}{\partial \tau} \Omega(\tau) d\tau \right\} \tag{20}$$

$$\sigma_z = \int_{0^-}^{t} J(t-\tau) \frac{\partial}{\partial \tau} v_0(\tau) d\tau - \frac{1}{2} \rho r^2 \int_{0^-}^{t} J(t-\tau) \frac{\partial}{\partial \tau} \Omega(\tau) d\tau \tag{21}$$

There are two unknown functions $v_0(t)$ and $A(t)$ which should be determined from boundary conditions.

3. A Pressurized Cylinder with Elastic Casing and Ablating Inner Surface

One application of the studies of viscoelastic cylinders is the stress analysis of solid-propellant grains in rockets under firing conditions. A typical simplified cross section is shown in Fig. 1. For simplicity of presentation, we assume that the spinning velocity $\omega(t)$ is zero. However, there is no additional difficulty in including this effect.

Since the inner radius is ablating, $a(t)$ is a monotonically increasing function of t. Thus the condition at the inner surface is

$$\sigma_r[a(t),t] = -\pi(t) \tag{22}$$

where $-\pi(t)$ is the internal pressure prescribed at the inner surface. The outer surface is encased and bonded to an elastic casing of thickness h. Assuming membrane theory, we obtain, from Rogers and Lee,[1]

$$\sigma_r[b,t] = -Be_\theta(b,t) \tag{23}$$

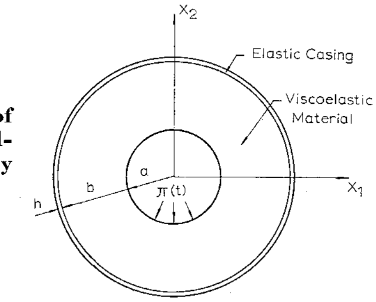

Fig. 1 Geometry of a viscoelastic cylinder reinforced by elastic casing.

where

$$1/B = [(1+\nu_R)/E_R][1 + (b/h)(1-\nu_R)] \tag{24}$$

E_R and ν_R are the Young's modulus and Poisson's ratio of the reinforcing elastic material.

Substituting (22) and (23) into (19), we obtain

$$\int_{0^-}^{t} [J(t-\tau) + K(t-\tau)] \frac{\partial}{\partial \tau} v_0(\tau) d\tau = \int_{0^-}^{t} [K(t-\tau) - B] \frac{\partial}{\partial \tau} \left[\frac{v_0(\tau)}{2} + \frac{A(\tau)}{2b^2} \right] d\tau \tag{25}$$

$$-\pi(t) = \int_{0^-}^{t} \left[J(t-\tau) + \frac{1}{2} K(t-\tau) \right] \frac{\partial}{\partial \tau} v_0(\tau) d\tau - \frac{1}{2a^2(t)} \int_{0^-}^{t} K(t-\tau) \frac{\partial}{\partial \tau} A(\tau) d\tau \tag{26}$$

Eliminating terms involving integrals of $v_0(t)$, we get

$$\int_{0^-}^{t} K(t-\tau) \frac{\partial}{\partial \tau} A(\tau) d\tau = \frac{Ba^2(t)\xi(t)}{a^2(t) - b^2} - \frac{2a^2(t) b^2 \pi(t)}{a^2(t) - b^2} \tag{27}$$

Eliminating terms involving integrals of $A(t)$ from (25) and (26), we get

$$\int_{0^-}^{t} \left[J(t-\tau) + \frac{1}{2} K(t-\tau) \right] \frac{\partial}{\partial \tau} v_0(\tau) d\tau = \frac{B\xi(t)}{2[a^2(t) - b^2]} - \frac{a^2(t)\, \pi(t)}{a^2(t) - b^2} \tag{28}$$

where

$$\xi(t) = v_0(t) b^2 + A(t) \tag{29}$$

Introducing an auxiliary material function $\phi(t)$ such that

$$\int_{0^-}^{t} [J(t-\tau) + K(t-\tau)] \frac{\partial}{\partial \tau} \phi(\tau) d\tau = \int_{0^-}^{t} [K(t-\tau) - B] \frac{\partial}{\partial \tau} \left[J(\tau) + \frac{K(\tau)}{2} \right] d\tau \tag{30}$$

Then, from (25) and (28), we obtain

$$B\xi(t) - \left(\frac{a^2(t)}{b^2} - 1 \right) \int_{0^-}^{t} \phi(t-\tau) \frac{\partial}{\partial \tau} \xi(\tau) d\tau = 2a^2(t)\pi(t) \tag{31}$$

According to Eqs. (27) and (28), the radial and tangential stress components are obtained in the form

$$\sigma_r = \frac{B\xi(t)}{2[a^2(t) - b^2]} \left[1 - \frac{a^2(t)}{r^2} \right] - \frac{a^2(t)\pi(t)}{a^2(t) - b^2} \left[1 - \frac{b^2}{r^2} \right] \tag{32}$$

$$\sigma_\theta = \frac{B\xi(t)}{2[a^2(t) - b^2]} \left[1 + \frac{a^2(t)}{r^2} \right] - \frac{a^2(t)\pi(t)}{a^2(t) - b^2} \left[1 + \frac{b^2}{r^2} \right] \tag{33}$$

The problem is then reduced to only one unknown function $\xi(t)$ which is governed by the integral Eq. (31). Removing

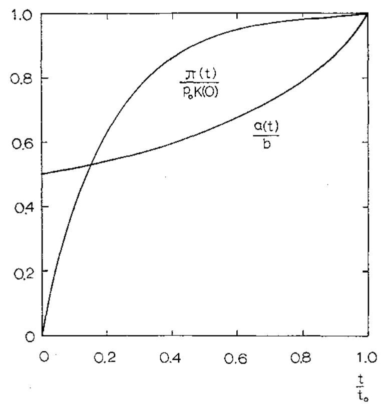

Fig. 2 Pressure and ablating functions.

the discontinuity at $t = 0$ and integrating by parts, give

$$\xi(t) - \int_{0^+}^{t} \phi^*(t,\tau)\xi(\tau)d\tau = \pi^*(t) \tag{34}$$

where

$$\phi^*(t,\tau) = [1/\phi_0(t)][1 - a^2(t)/b^2](\partial/\partial\tau)\phi(t-\tau)$$

$$\pi^*(t) = [1/\phi_0(t)][2a^2(t)\pi(t)]$$

$$\phi_0(t) = B + [1 - a^2(t)/b^2]\phi(0)$$

Equation (34) is a typical Volterra integral equation of the second kind. The general solution has the form (see Volterra[4]),

$$\xi(t) = \pi^*(t) + \int_{0^+}^{t} Q^*(t,\tau)\pi^*(\tau)d\tau \tag{35}$$

where Q^* is the resolvent of Eq. (34).

The iterative form of Q^* can be found as

$$Q^*(t,\tau) = \sum_{n=1}^{\infty} q^{(n)}(t,\tau) \tag{36}$$

where

$$q^{(1)}(t,\tau) = \phi^*(t,\tau)$$

$$q^{(n)}(t,\tau) = \int_{\tau}^{t} q^{(1)}(t,s)q^{(n-1)}(s,t)ds$$

The first-order approximation is thus

$$\xi^{(1)}(t) = \pi^*(t) + \int_{0^+}^{t} \phi^*(t,\tau)\pi^*(\tau)d\tau \tag{37}$$

The expressions for e_r and e_θ are more complicated. Both $A(t)$ and $v_0(t)$ should be found explicitly. Introducing a function $G(t)$ defined by

$$\int_{0^-}^{t} K(t-\tau)\frac{\partial}{\partial\tau}G(\tau)d\tau = 1 \tag{38}$$

From (27), we obtain

$$A(t) = \int_{0^-}^{t} G(t-\tau)\frac{\partial}{\partial\tau}\left[\frac{Ba^2(\tau)\xi(\tau)}{a^2(\tau)-b^2} - \frac{2a^2(\tau)b^2\pi(\tau)}{a^2(\tau)-b^2}\right]d\tau \tag{39}$$

Corresponding volume change $v_0(t)$ is when

$$v_0(t) = \frac{1}{b^2}[\xi(t) - A(t)] \tag{40}$$

Hence,

$$e_r = \xi(t)/2b^2 - [A(t)/2b^2][1 + b^2/r^2] \tag{41}$$

$$e_\theta = \xi(t)/2b^2 - [A(t)/2b^2][1 - b^2/r^2] \tag{42}$$

Two special cases are discussed

1) The material is assumed to be incompressible; then,

$$\sigma_{r,\theta} = -p(t) + \int_{0^-}^{t} K(t-\tau)\frac{\partial}{\partial\tau}e_{r,\theta}(\tau)d\tau \tag{43}$$

$$v_0(t) = 0 \tag{44}$$

$$J(t) \to \infty$$

where $-p(t)$ is the hydrostatic pressure, and

$$\xi(t) = A(t) \tag{45}$$

$$-e_r = e_\theta = A(t)/2r^2 \tag{46}$$

Equation (28) gives

$$-p(t) = \frac{BA(t)}{2[a^2(t)-b^2]} - \frac{a^2(t)\pi(t)}{a^2(t)-b^2} \tag{47}$$

Equation (27) gives

$$BA(t) + \left[\frac{b^2}{a^2(t)} - 1\right]\int_{0^-}^{t} K(t-\tau)\frac{\partial}{\partial\tau}A(\tau)d\tau = 2b^2\pi(t) \tag{48}$$

and then,

$$\sigma_r = \frac{BA(t)}{2[a^2(t)-b^2]}\left[1 - \frac{a^2(t)}{r^2}\right] - \frac{a^2(t)\pi(t)}{a^2(t)-b^2}\left[1 - \frac{b^2}{r^2}\right] \tag{49}$$

$$\sigma_\theta = \frac{BA(t)}{2[a^2(t)-b^2]}\left[1 + \frac{a^2(t)}{r^2}\right] - \frac{a^2(t)\pi(t)}{a^2(t)-b^2}\left[1 + \frac{b^2}{r^2}\right] \tag{50}$$

The solution of this particular case has been discussed extensively by Shinozuka.[5]

2) The inner surface is assumed to be nonablating; then,

$$a(t) = a = \text{const}$$

Integral Eq. (34) is reduced to a convolutional type

$$\xi(t) - \int_{0^+}^{t} \phi^*(t-\tau)\xi(\tau)d\tau = \pi^*(t) \tag{51}$$

where

$$\phi^*(t-\tau) = 1/\phi_0(1 - a^2/b^2)(\partial/\partial\tau)\phi(t-\tau)$$

$$\pi^*(t) = 2a^2/\phi_0\pi(t)$$

$$\phi_0 = B + \phi(0)(1 - a^2/b^2)$$

Analytical solution can be obtained easily by using Laplace transform techniques.

4. Numerical Examples

The primary purpose of our presenting numerical examples is to show the accuracy of the iterated analytical solution. The results obtained by directly solving the governing integral Eq. (34) and the results obtained by evaluating the first-order approximation of analytical solution, (37), are compared. The usual finite-difference technique[6] is used to solve the integral equation.

It should be noted that the numerical calculations here are much simpler than the previous reports on the same subject, since not only the formulation of this problem is reduced to only a single integral equation, but also the stress components are expressed as simple algebraic relations of the unknown function.

For a particular problem, the ablating inner surface is taken as a simple function

$$a^2(0)/a^2(t) = 1 - [1 - a^2(0)/b^2]t/t_0 \tag{52}$$

The pressure acting on the moving inner surface is an ex-

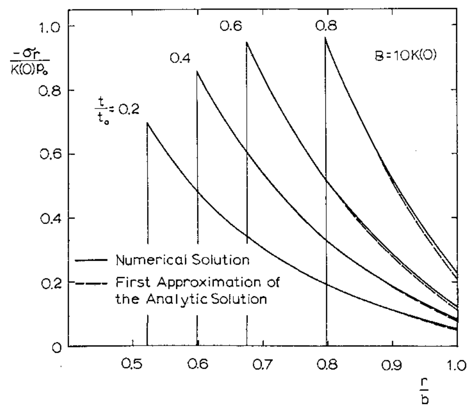

Fig. 3 Radial stress distribution in cylinder.

ponentially increasing function

$$\pi(t)/K(0) = p_0[1 - e^{-mt/t_0}] \tag{53}$$

where m is chosen to be 5. Dimensionless quantities $a(t)/b$ and $\pi(t)/K(0)p_0$ are shown in Fig. 2. The material chosen for calculation is a viscoelastic carboxy terminated polymer, commercially produced as unfilled HC rubber. Typical data in tension were closely fitted to a three-element model by Williams[7] as

$$E(t) = E_e\{1 + (E_g/E_e - 1)e^{-t/t_0}\} \tag{54}$$

where t_0 is the characteristic time of the material considered. As a typical example, we consider the specific values

$$E_e = 123 \text{ psi}$$

$$E_g = 12500 \text{ psi}$$

The dilatational effect is considered to be elastic with initial Poisson's ratio

$$\nu_0 = \tfrac{1}{3} \quad \text{at} \quad t = 0$$

The corresponding relaxation functions $J(t)$ and $K(t)$ are then

$$K(t) = \frac{2E_g}{3(E_g/E_e) - (1 - 2\nu_0)} \left\{1 + \frac{3}{2(1 + \nu_0)} \times \left(\frac{E_g}{E_e} - 1\right) \exp - \frac{3 - (1 - 2\nu_0)(E_e/E_g)}{2(1 + \nu_0)} \frac{t}{t_0}\right\}$$

$$J(t) = \frac{E_g}{3(1 - 2\nu_0)} - \frac{1}{3} K(t) \tag{55}$$

The initial value of $K(t)$ at time $t = 0$ is

$$K(0) = E_g/(1 + \nu_0) \tag{56}$$

Now the auxiliary material function $\phi(t)$ can be obtained

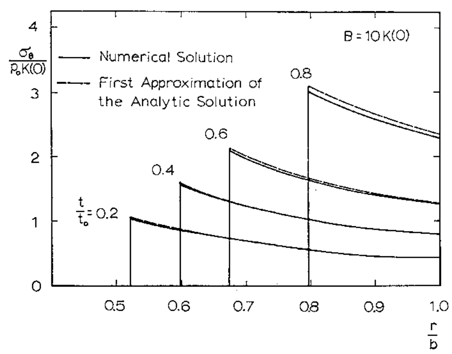

Fig. 4 Circumferential stress distribution in cylinder.

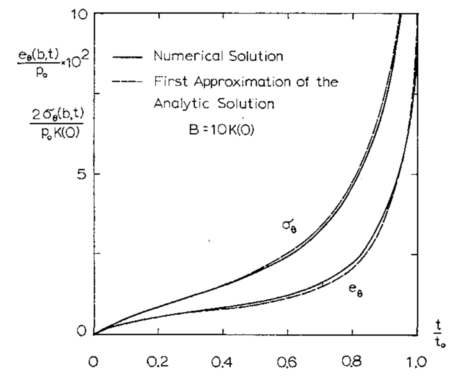

Fig. 5 History at interface.

easily through Eq. (30) by applying the Laplace transform technique. Functions $\phi_0(t)$, $\phi^*(t,\tau)$, and $\pi^*(t)$ are then given in closed form for this particular model material.

The stiffness of the elastic casing varies from

$$B/K(0) = 10, \tfrac{50}{3}, 50, 100$$

The stress and strain distributions for this particular problem are shown in Figs. 3–5. It shows that the error of the first-order approximation is less than 3% at all times. Figure 6 shows the effect of elastic casings having different stiffnesses. As discussed by Rogers and Lee,[1] this effect is dominant.

For problems using general measured material functions, the material function $\phi(t)$ can be computed through Eq. (30) by simple finite-difference numerical integrations. The rest of the procedures can be followed without further difficulty.

5. Conclusions

Linear viscoelastic solutions are given for a pressurized hollow cylinder bonded to an elastic casing. We have indicated that this simple alternative procedure has simplified the mathematical analysis to a single integral equation for solving one unknown function $\xi(t)$. The stress components can then be expressed as algebraic relations of $\xi(t)$ and surface pressure $\pi(t)$. A numerical example also proves that the first-order analytical solution gives a good approximation which might have some applications in the stress estimation of propellant designs.

It might be interesting to note that the function $\xi(t)$ is proportional to deformation of the elastic casing $u(b,t)$. If $u(b,t)$ can be measured experimentally for a particular time t, the stress distributions can then be evaluated for that instant without knowing the whole history, since they are algebraically related.

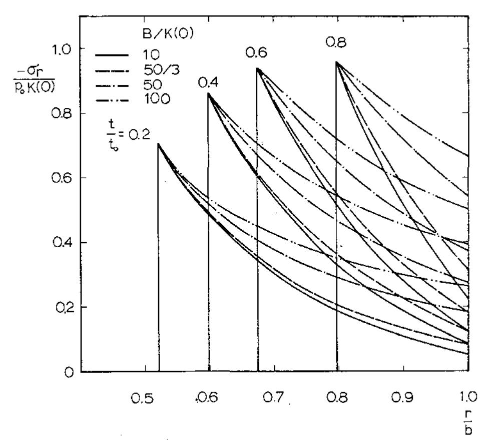

Fig. 6 Influence of elastic casings.

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Buckling of Elliptic Cylinders under Normal Pressure

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An analytical and experimental investigation was conducted to determine the buckling behavior of elliptic cylinders under normal pressure. Tests were performed on a total of 80 models made of polyvinyl-chloride sheet. The models were simply supported and the buckling mode was found as an oval dimple which appears around the surface with the least curvature. Theoretical buckling pressure was also obtained by using Goldenveizer's linear, thin-shell theory in conjunction with Galerkin's method for solution. The test data and theory agree fairly well, provided that the shell is not too short or too thick. For design purposes, curves are provided in nondimensional form to show the interaction of the physical quantities of the cylinders and the buckling pressures obtained by theory as well as by experiment.

Nomenclature

a	= semimajor axis of ellipse
b	= semiminor axis of ellipse
d	= half of periphery of ellipse
dA	= surface element
E	= Young's modulus
l	= length
M_x	= moment at ends due to buckling displacements
n	= circumferential wave number
p	= normal pressure
R	= radius of curvature of ellipse
r	= a^2/b (radius of curvature of ellipse at its minor axis)
s	= β/l
t	= thickness
T_x'	= axial membrane force at the ends due to buckling displacements
$T_x, T_x,$ T_{xa}, T_{sx}	= prebuckling membrane forces (Fig. 2)
u	= u'/t
v	= v'/t
w	= w'/t
u', v', w'	= buckling displacements along α, β, and z, directions: w' is positive if inward
x	= α/l
α, β, z	= curvilinear coordinate system (Fig. 1)
λ	= $(1 - \nu^2)(l/t)(p/E)$(buckling parameter)
ν	= Poisson's ratio
ϕ	= angle between normal and semiminor axis (Fig. 3)

Introduction

THE strongest and yet the lightest shell structure for bearing normal external pressure is undoubtedly a sphere; however, the requirement for a structure to perform functions other than carrying pressure often forces us to consider other shell configurations. The cylindrical shell of elliptic cross section is one such configuration.

The present paper is concerned with the buckling of elliptical cylinders with simply supported ends. This problem has been considered before by B. I. Slepov.[1] However, it appeared necessary to reopen the investigation for the following reasons: 1) Slepov expressed the buckling mode as a Fourier series but used only one term in the analysis, which is too crude an approximation for good accuracy. 2) Slepov's theoretical analysis was supported by only two test points, which is insufficient to confirm the theory or establish any trend. 3) no design criterion incorporating the deviation between theoretical and test results has yet been established.

In this investigation, a total of 80 models made of polyvinyl-chloride (PVC) sheet were buckled in a test program carried out by the second author. The analytical work was performed by the first author, who used Goldenveizer's thin-shell theory[2] and Galerkin's method for solution. The buckling prediction made from this investigation is presented in graphical form with a wide range of geometric and material parameters.

Theory

We chose the coordinate system α and β along the lines of curvature on the middle surface of the shell and z the normal to the middle surface as shown in Fig. 1. Let t and l be the wall thickness and length of the shell, respectively, and let u', v', and w' be the buckling displacements of a point on the middle surface along the α, β, and z directions, respectively, with w' considered as positive if inward. Accordingly, we adopted the following nondimensional quantities in the sequel:

$$x = \alpha/l, \quad s = \beta/l, \quad u = u'/t \qquad v = v'/t, \quad w = w'/t \tag{1}$$

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